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On Bounded Multilinear Forms on a Class of l^p Spaces

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In this paper we generalize an old result of Littlewood and Hardy about bilinear forms defined in a class of sequence spaces. Historically, Littlewood [*Quart. J. Math.* **1** (1930)] first proved a result on bilinear forms on bounded sequences and this result was then generalized by Hardy and Littlewood in a joint paper [*Quart. J. Math.* 5(1934)] to bilinear forms on a class of l^p spaces. Later Davie and Kaijser proved Littlewood's results for multilinear forms. In this paper, Theorems A and B generalize the results to multilinear forms on l^p spaces. All the results are stated at the end of Section 1. Theorems A and B are proved, respectively, in Sections 2 and 3.

1. Introduction, Notation and Results

In a paper written in 1930, [5], Littlewood proved a necessary condition on the coefficients of a bilinear form on l^∞ which may be stated as: If

$$A: l^\infty \times l^\infty \to \mathbb{C}; \quad A(x, y) = \sum_i \sum_j a_{ij} x_i y_j$$

is continuous and bounded by M then

$$\sum_i \left(\sum_j |a_{ij}|^2 \right)^{1/2} \leq KM \qquad (i \text{ and } j \text{ may be interchanged}), \tag{1.1.1}$$

$$\sum_i \sum_j |a_{ij}|^\mu \leqslant (KM)^\mu \qquad \mu = 4/3. \tag{1.1.2}$$

This result has been generalized by Littlewood and Hardy [2] in a joint paper written in 1934 to get (1.11), below, which in modern terminology can be stated as

$$l^p(D_1) \hat{\otimes}_\epsilon l^q(D_2) \subset l^\mu(D_1 \times D_2), \tag{1.2}$$

where $\mu = 4/(3 - 2(1/p + 1/q))$, D_i are discrete.

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409/81/2 18

Davie and Kaijser [1, 4] generalized (1.1) for n-linear forms, proving theorem (1.12) below which can be stated also as

$$_{\epsilon}\hat{\otimes}_{i=1}^{n} l^1(D_i) \subset l^r(D), \tag{1.3}$$

where $r = 2n/(n+1)$ and $D = D_1 \times \cdots \times D_n$, D_i is discrete.

The method and the notation we were able to develop in this paper show that (1.2) comes directly from (1.1) and then using (1.3) we are able to generalize (1.2) to n-tensors.

The paper of Hardy and Littlewood [2] is quite hard to read. We believe that this presentation is simpler, even though the idea of the proof is the same. One of the central difficulties was the notation. The number μ which appears in connection with (1.2) indicates the need of a new notation.

For a collection of sequence spaces $l^{p_1},..., l^{p_n}$ let us define the index

$$\alpha_i = 1/p_i \tag{1.4}$$

and the multi-index

$$\alpha = (\alpha_1,..., \alpha_n) \in [0, 1]^n. \tag{1.5}$$

Very often we are using multi-indices $\beta \in \mathbb{N}^n$ to perform summations and then we shall denote

$$\dot{\beta}_i \in \mathbb{N}^{n-1} \tag{1.6}$$

the multi-index obtained by letting the coordinate β_i drop away.

If $\alpha \in [0, 1]^n$ we are going to use the notations:

$$\alpha_i \text{ to represent its } i\text{th coordinate,} \tag{1.7}$$

$$\alpha'_i \text{ to represent the } i\text{th coordinate of the dual multi-index } \alpha' \text{ that is } \alpha'_i = 1 - \alpha_i, \tag{1.7.1}$$

$$|\alpha| = \sum_{i=1}^{n} \alpha_i = \text{order of the multi-index } \alpha, \tag{1.7.2}$$

$$(\alpha) = l^{p_1} \times \cdots \times l^{p_n} \text{ the cartesian product of the family of sequences spaces corresponding to } \alpha. \tag{1.7.3}$$

$$x \in (\alpha), \qquad x = (x_\beta)_\beta = (x_{\beta_1},..., x_{\beta_n})_\beta. \tag{1.8}$$

Throughout this paper we are dealing with functions defined on (α) for some $\alpha \in [0, 1]^n$ which are n-linear, continuous and $\mathbb{C}$-valued. The notation will be

$$A: (\alpha) \longrightarrow \mathbb{C},$$
$$A: x = (x_\beta)_\beta \curvearrowright \sum_{\beta \in \mathbb{N}^n} a_\beta x_\beta, \tag{1.9}$$

where $(a_\beta)_\beta$ denotes the matrix of A.

A basic tool in our calculations will be partial sums on the entries of A which we shall denote

$$S_{\beta_i} = \left(\sum_{\beta_i} |a_\beta|^2\right)^{1/2}. \tag{1.10}$$

As an illustration of the use of this notation we present the theorem of Hardy and Littlewood which will be the object of generalization in this paper followed by the result of Davie–Kaijser:

(1.11) THEOREM (Littlewood and Hardy). *Let* $\alpha = (\alpha_1, \alpha_2) \in [0, 1]^2$; $|\alpha| \leqslant 1$, A: $(\alpha) \to \mathbb{C}$ *a bilinear form bounded by* M; $A(x) = \sum_{\beta \in \mathbb{N}^2} a_\beta x_\beta$. Then,

(a) $\sum_\beta S_\beta^\lambda \leqslant (KM)^\lambda$; $\lambda = 1/(1 - |\alpha|)$, $i = 1, 2$.

(b) *If* $|\alpha| \leqslant \frac{1}{2}$ *then* $(\sum_\beta |a_\beta|^\mu)^{1/\mu} \leqslant KM$; $\mu = 4/(3 - 2|\alpha|)$.

(c) *If* $1 \geqslant |\alpha| \geqslant \frac{1}{2}$ *and* $(\forall i)(\alpha_i \leqslant \frac{1}{2})$ *then* $(\sum |a_\beta|^\lambda)^{1/\lambda} \leqslant KM$, λ *as above.*

(1.12) THEOREM (Davie and Kaijser). *If* $\alpha = (0,..., 0) \in [0, 1]^n$ *and* A: $(\alpha) \to \mathbb{C}$; $A(x) = \sum_{\beta \in \mathbb{N}^n} a_\beta x_\beta$ *is bounded by* M, *then*

(a) $\sum_{\beta_i} S_{\beta_i} \leqslant KM$, $i = 1, 2,..., n$.

(b) $(\sum_\beta |a_\beta|^r)^{1/r} \leqslant KM$, $r = 2n/(n + 1)$.

In this paper we shall prove the following two theorems:

(1.13) THEOREM A. *If* $\alpha = (\alpha_1,..., \alpha_n) \in [0, 1]^n$; $|\alpha| \leqslant \frac{1}{2}$ *and* A *is a multilinear form bounded by* M, A: $(\alpha) \to \mathbb{C}$; $A(x) = \sum_\beta a_\beta x_\beta$, *then we have*

$$\sum_{\beta_i} S_{\beta_i}^\lambda \leqslant (KM)^\lambda,$$

where $\lambda = 1/(1 - |\alpha|)$.

(1.14) THEOREM B. *If* A, α *and* λ *are as in Theorem* A, *then*

$$\left(\sum_\beta |a_\beta|^\mu\right)^{1/\mu} \leqslant KM; \qquad \mu = \frac{2n}{n + 1 - 2|\alpha|}.$$

The constant K is the same from (1.12) through (1.15), in [4] $K \leqslant 2^{(n-1)/2}$, this is the form of K in (1.11) too. It is a consequence of Khintchines inequality in the proof of Littlewood's result by Davie and Kaijser.

If $|\alpha| > \frac{1}{2}$ let us denote by $\dot{\alpha}_i$ the multi-index obtained by replacing α_i by zero. Then, if $|\dot{\alpha}_i| \leqslant \frac{1}{2}$, we have

$$\sum_{\beta_i} S_{\beta_i}^{\lambda} \leqslant (KM)^{\lambda}; \qquad \lambda = \frac{1}{1-|\alpha|} \tag{1.15}$$

and this shows that the theorem of Hardy and Littlewood follows immediately from our theorems.

2. Proof of Theorem A

The proof is by induction.

(2.1) Induction Hypothesis. *We suppose that the theorem is true for*

$$\alpha' = (\alpha_1, \ldots, \alpha_{k-1}, 0, \ldots, 0).$$

(2.2) Lemma (Hölder's converse). *If* $(x_i)_i \in l^r$ *is of norm* 1 *and* $(a_i)_i$ *is such that* $\sum_i x_i a_i \leqslant C$ *where* C *is a constant, then*

$$\sum_i a_i^{r'} \leqslant C^{r'},$$

where $1/r' = 1 - 1/r$.

Now, if we define $A'\colon (\alpha') \to \mathbb{C}$; $A'(z) = A(zx_{\beta_k})$ for some $(x_{\beta_k})_{\beta_k}$ of norm 1 fixed, then we have

(2.3) Lemma. $\sum_{\beta_k} (\sum_{\beta_k} |a_\beta|^2 \, |x_{\beta_k}|^2)^{\lambda'/2} \leqslant (KM)^{\lambda}$ *with* $\lambda' = 1/(1-|\alpha'|)$.

Proof. By the induction hypothesis.

Now the kth inequality follows from (2.3) and (2.2) with $1/r = \alpha_k \lambda'$ (and hence $r' = \lambda/\lambda'$):

$$\sum_{\beta_k} S^{\lambda} \leqslant (KM)^{\lambda}. \tag{2.4.k}$$

To prove the other relations, for $i \neq k$, we need the lemma:

(2.5) LEMMA.

$$\sum_{\beta_k}\left(\sum_{\hat{\beta}_k}\frac{|a_\beta|^2}{S_{\beta_i}^{2-\lambda'}}\right)^{\lambda/\lambda'} \leqslant (KM)^\lambda$$

with $i \neq k$ and λ' and λ as above.

Proof. This follows directly from the following lemma combined with Lemma (2.2) with $1/r = \alpha_k \lambda'$.

(2.6) LEMMA.

$$\sum_{\beta_k}\sum_{\hat{\beta}_k}\frac{|a_\beta|^2}{S_{\beta_i}^{2-\lambda'}}|x_{\beta_k}|^{\lambda'} \leqslant (KM)^{\lambda'},$$

where λ' is as above and $(x_{\beta_k})_{\beta_k}$ is of norm 1.

Proof.

$$\sum_{\beta_k}\sum_{\hat{\beta}_k}\frac{|a|_\beta^2}{S_{\beta_i}^{2-\lambda'}}|x_{\beta_k}|^{\lambda'} = \sum_{\beta_i}\sum_{\hat{\beta}_i}\frac{|a_\beta|^2}{S_{\beta_i}^{2-\lambda'}}|x_{\beta_k}|^{\lambda'}$$

$$= \sum_{\beta_i}\sum_{\hat{\beta}_i}\frac{|a_\beta|^{2-\lambda'}}{S_{\beta_i}^{2-\lambda'}}|a_\beta|^{\lambda'}|x_{\beta_k}|^{\lambda'}$$

$$\leqslant \sum_{\beta_i}\left(\sum_{\hat{\beta}_i}\frac{|a_\beta|^2}{S_{\beta_i}^2}\right)^{(2-\lambda')/2}\left(\sum_{\hat{\beta}_i}|a_\beta|^2|x_{\beta_k}|^2\right)^{\lambda'/2} = A,$$

by Hölder's inequality with

$$\frac{2-\lambda'}{2}+\frac{\lambda'}{2}=1, \qquad 0<\frac{\lambda'}{2}<1.$$

And as $\sum_{\hat{\beta}_i}(|a_\beta|^2/S_{\beta_i}^2) = 1$ we have

$$A = \sum_{\beta_i}\left(\sum_{\hat{\beta}_i}|a_\beta|^2|x_{\beta_k}|^2\right)^{\lambda'/2} \leqslant (KM)^{\lambda'}$$

by the induction hypothesis, proving (2.6).

Now, for $i \neq k$ we have

$$\sum_{\beta_i} S_{\beta_i}^{\lambda} = \sum_{\beta_i} S_{\beta_i}^{\lambda-2}\sum_{\hat{\beta}_i}|a_\beta|^2 = \sum_{\beta_k}\sum_{\hat{\beta}_k}\frac{|a_\beta|^2}{S_{\beta_i}^{2-\lambda}}$$

$$= \sum_{\beta_k}\sum_{\hat{\beta}_k}\frac{|a_\beta|^{2r}}{S_{\beta_i}^{2-\lambda}}|a_\beta|^{2r'} = B.$$

Put now

$$r = \frac{2-\lambda}{2-\lambda'} < 1,$$
$$r' = 1 - r = \frac{\lambda - \lambda'}{2-\lambda'} < 1 \tag{2.7}$$

and then by Hölder's inequality

$$B \leqslant \sum_{\beta_k} \left(\sum_{\beta_k} \frac{|a_\beta|^2}{S_{\beta_l}^{2-\lambda'}} \right)^r \left(\sum_{\beta_k} |a_\beta|^2 \right)^{r'}$$
$$\leqslant \left(\sum_{\beta_k} \left(\sum_{\beta_k} \frac{|a_\beta|^2}{S_{\beta_l}^{2-\lambda'}} \right)^{r/s} \right)^s \left(\sum_{\beta_k} \left(\sum_{\beta_k} |a_\beta|^2 \right)^{r'/s'} \right)^{s'} = C$$

by Hölder's inequality with $s + s' = 1$.

Then, by choosing s so that $r/s = \lambda/\lambda'$ it turns out that $r'/s' = \lambda/2$ and we have

$$C \leqslant (KM)^{s\lambda}(KM)^{s'\lambda} = (KM)^\lambda.$$

(2.8) Proposition. *If $\frac{1}{2} \leqslant |\alpha| < 1$ but $|\dot{\alpha}_l| \leqslant \frac{1}{2}$ then*

$$\sum_{\beta_l} S_{\beta_l}^\lambda \leqslant (KM)^\lambda; \qquad \lambda = \frac{1}{1-|\alpha|}.$$

Proof. From Lemmas 2.3 and 2.2.

3. Proof of Theorem B

The proof is essentially the same as in the bilinear case. It is performed by using Hölder's inequality to split $\sum_\beta |a_\beta|^\mu$ into several factors (in n factor in case of n-linear forms). Later using the Minkowski inequality we are able to use inequalities in Theorem A to make the final majoration. We shall do the proof for $n = 3$, the general case being obvious.

$$\sum_\beta |a_\beta|^\mu = \sum_\beta |a_\beta|^{\mu_1+\mu_2+\mu_3}$$
$$\leqslant \prod_{l=1}^{3} \left(\sum_{\beta_1} \left(\sum_{\beta_2} \left(\sum_{\beta_3} |a_\beta|^{\mu_l/2s_l^3} \right)^{s_l^3/s_l^2} \right)^{s_l^2/s_l^1} \right)^{s_l^1} = \prod_{l=1}^{3} A_l \tag{3.1}$$

by applying Hölder's inequality with the multi-indices

$$s^i = (s_1^i, s_2^i, s_3^i); \qquad s_1^i + s_2^i + s_3^i = 1 \tag{3.2}$$

$i = 1, 2, 3$.

Now, by successive application of Minkowski inequality in the second and third factors (once and twice, respectively) we shall be able to majorize them via Theorem A each one beginning with the right index. The whole is subordinated to the system of equations

$$\frac{\mu_1}{s_1^3} = \frac{\mu_2}{s_2^3} = \frac{\mu_3}{s_3^2} = 2, \tag{1}$$

$$\begin{aligned} s_1^3 &= s_1^2; & 2s_1^2 &= \lambda s_1^1, \\ s_2^3 &= s_2^1; & 2s_2^1 &= \lambda s_2^2, \\ s_3^2 &= s_3^1; & 2s_3^1 &= \lambda s_3^3, \end{aligned} \tag{2}$$

$$(\forall i) \qquad \left(\sum_{k=1}^{3} s_k^i = 1 \right). \tag{3}$$

From (2) and (3) we may conclude that a given multi-index has all coordinates equal to a but one which is $(2/\lambda)a$:

$$\begin{aligned} s^1 &= \left(a, a, \frac{2}{\lambda} a\right); \qquad a = \frac{\lambda}{2(\lambda+1)}, \\ s^2 &= \left(a, \frac{2}{\lambda} a, a\right), \\ s^3 &= \left(\frac{2}{\lambda} a, a, a\right), \end{aligned}$$

and $\mu_1 = \mu_2 = \mu_3 = 2a = \lambda/(\lambda+1)$.

In this way we have (see [3, p. 32]),

$$A_1 = \left(\sum_{\beta_1} \left(\sum_{\beta_2} \sum_{\beta_3} |a_\beta|^2 \right)^{\lambda/2} \right)^{1/\lambda+1} \leqslant (KM)^{\lambda/\lambda+1},$$

$$\begin{aligned} A_2 &= \left(\sum_{\beta_1} \left(\sum_{\beta_2} \left(\sum_{\beta_3} |a_\beta|^2 \right)^{\lambda/2} \right)^{2/\lambda} \right)^{\lambda/2(\lambda+1)} \\ &\leqslant \left(\sum_{\beta_2} \left(\sum_{\beta_1} \sum_{\beta_3} |a_\beta|^2 \right)^{\lambda/2} \right)^{1/\lambda+1} \leqslant (KM)^{\lambda/\lambda+1}, \end{aligned}$$

$$A_3 = \left(\sum_{\beta_1}\sum_{\beta_2}\left(\sum_{\beta_3}|a_\beta|^\lambda\right)^{2/\lambda}\right)^{\lambda/2(\lambda+1)}$$

$$\leqslant \left(\sum_{\beta_1}\left(\sum_{\beta_3}\left(\sum_{\beta_2}|a_\beta|^2\right)^{\lambda/2}\right)^{2/\lambda}\right)^{\lambda/2(\lambda+1)}$$

$$\leqslant \left(\sum_{\beta_3}\left(\sum_{\beta_1}\sum_{\beta_2}|a_\beta|^2\right)^{\lambda/2}\right)^{1/\lambda+1} \leqslant (KM)^{\lambda/\lambda+1},$$

$$\prod_{i=1}^{3} A_i \leqslant (KM)^{3\lambda/\lambda+1} = (KM)^\mu.$$

By substitution of λ we have $\mu = 3/(2-|\alpha|)$ or, in general, in the case of n-linear forms

$$\mu = \frac{2n}{n+1-2|\alpha|}.$$

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